

# Asymmetric Orbifolds and Higher Level Models

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## Abstract

I introduce a class of string constructions based on asymmetric orbifolds leading to level two models. In particular, I derive in detail various models with gauge groups  $E_6$  and  $SO(10)$ , including a four generation  $E_6$  model with two adjoint representations. The occurrence of multiple adjoint representations is a generic feature of the construction. In the course of describing this approach, I will address the problem of twist phases in higher twisted sectors of asymmetric orbifolds.

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# 1 Introduction

The apparent unification of gauge couplings [1] in the context of minimal supersymmetry when extrapolated to high energies has created growing interest in supersymmetric Grand Unified Theories and Superstring Theories. The unification scale is impressively close to but slightly lower than the string scale [2]. From a string theorist point of view, it is presently unclear whether this is indicative for an intermediate Grand Unification (GUT) group or rather for string effects such as threshold [2, 3] or strong coupling effects [4].

If one chooses to invoke a GUT group such as  $SO(10)$ , it is necessary to construct string models based on level  $k > 1$  Kac-Moody algebras in order to potentially obtain Higgs representations able to break the gauge group in a satisfactory way. This, however, is rather awkward and most of the string models constructed so far are typically realized at level 1. Moreover, all known leptons and quarks are either singlets or in the fundamental representations of  $SU(3)$  and  $SU(2)$  which are already available at level 1. Similarly, non-standard Higgs representations such as triplets are strongly constrained as they yield tree-level contributions to the  $\rho_0$ -parameter<sup>1</sup>. Indeed, through high precision measurements,  $\rho_0$  is now known to be very close to unity<sup>2</sup> [5],

$$\rho_0 = 0.9985 \pm 0.0019^{+0.0012}_{-0.0009}. \quad (1)$$

Although all this seems to encourage the construction of level 1 models and consequently the rejection of a simple intermediate GUT group, in such a case one necessarily encounters phenomenologically problematic fractionally charged particles<sup>3</sup>. This important statement has been made precise by Schellekens [8]: any level 1 compactification of the heterotic string with the GUT scale normalization of  $\sin^2 \theta_W = 3/8$  has either fractional charges<sup>4</sup> or an enhanced gauge group containing  $SU(5)$ .

The remaining options are either to accept fractional charges and to make them sufficiently heavy and rare [10] or to confine them through an extra non-Abelian gauge group [11], or alternatively, to proceed to higher levels  $k$ . The latter is the option chosen in this work. Since higher level models generically (though not always) possess adjoint representations, it is natural to use them for GUT breaking. However, this is not the only possibility and one may use adjoints in a rather different way: it has been noted that the addition of a color-octet (iso-singlet) and an iso-triplet (color-singlet) to the minimal supersymmetric standard model (MSSM) can lead to gauge coupling unification at the correct (string) scale with a lower  $\alpha_s$  (as found in most low-energy determinations), when the masses of both extra multiplets are a few times  $10^{12}$  GeV [12].

Higher level models were discussed systematically by Lewellen [13] who also presented some level 2 examples using free fermions. The fermionic construction was then further exploited

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<sup>1</sup>By definition,  $\rho_0$  describes new physics contributions to the  $\rho$  parameter so that in the Standard Model with its minimal Higgs sector  $\rho_0 \equiv 1$ .

<sup>2</sup>This value was obtained allowing even extra parameters describing non-standard loop contributions to the vector boson self-energies ( $S$ ,  $T$  and  $U$ ) and non-standard couplings of the  $Z$  boson to  $b$  quarks.

<sup>3</sup>All Standard Model particles have integral charges in the sense that if they transform in the singlet (triplet, antitriplet) congruency class of  $SU(3)_C$  then they have electric charge 0 (2/3, 1/3) mod  $\mathbf{Z}$ .

<sup>4</sup>As has been noted in Reference [9], there remains the logical possibility that the fractionally charged states appear only at massive levels, but no examples are known.

in References [9, 14]. Level 1 models with gauge groups  $G \times G$ , which can be broken to the diagonal  $G$  by turning on flat directions were studied by Finnell [15], who found three generation  $SU(5)$  models of this type. The diagonal  $G$  is then expected [16] to be realized at level 2.

Various methods in the context of symmetric orbifolds were introduced in Reference [16] and recently followed up [17]. In the present paper, I focus on asymmetric  $Z_2 \times Z_N$  orbifolds which lead to level 2 Kac-Moody algebras. This includes, in particular, models with gauge group  $E_6$ . This group has the unique property of being able to accommodate each fermion generation in its chiral fundamental representation.  $E_6$  string GUT models have not been constructed before. They have the special feature that no unwanted exotic representations can occur in the massless spectrum. In the construction here, the exceptional groups appear quite naturally: the simplest version of this construction (see section 3) yields  $E_8$  and  $N = 2$  supersymmetry (or simple supersymmetry in  $D = 6$ ); the simplest  $N = 1$  model has gauge group  $E_7$ ; and among the simplest *chiral* possibilities is  $E_6$ .

This article is organized as follows: Section 2 summarizes some facts about higher level string models. In section 3, I discuss the relevant aspects of asymmetric orbifolds and introduce the basic strategy how to obtain models at level 2. Section 4 describes  $Z_2 \times Z_3$  orbifolds yielding  $E_6^{k=2}$  gauge groups. I will, in particular, discuss in detail new issues arising in higher twisted sectors of *non-prime* asymmetric orbifolds which were not worked out in the original article on asymmetric orbifolds by Narain, Sarmadi and Vafa [18].  $SO(10)$  and  $E_6$  models based on  $Z_2 \times Z_4$  orbifolds are the subject of section 5. Here I show how to avoid the phase ambiguities of the type encountered in section 4. I present my conclusions in section 6.

## 2 Higher level string models

There are three basic relations [19] of relevance to higher level string model building. The first one,

$$c = \frac{k \dim G}{k + \tilde{h}}, \quad (2)$$

relates the central extension of the Kac-Moody algebra being proportional to the level  $k$  to its contribution to the conformal anomaly  $c$ , which in turn parametrizes the central extension of the Virasoro algebra. In Eq. (2)  $\dim G$  is the dimension of the gauge group  $G$  and  $\tilde{h}$  is the dual Coxeter number. The second relation,

$$h_R = \frac{C_R}{2(k + \tilde{h})}, \quad (3)$$

gives the conformal dimension of a primary field transforming in representation  $R$  under  $G$ .  $C_R$  is the quadratic Casimir invariant of  $R$ ,

$$\text{tr}_{R_1} F^2 = \frac{C_1 \dim R_1}{C_2 \dim R_2} \text{tr}_{R_2} F^2, \quad (4)$$

where  $F^2$  refers to the gauge field kinetic energy and for the adjoint representation  $A$  one has  $C_A = 2\tilde{h}$ . Finally, there is an inequality restricting the (unitary) representations  $R$  in which primary fields can transform for given  $k$ ,

$$k \geq \sum_{i=1}^{\text{rank } G} n_i m_i. \quad (5)$$

Here  $n_i$  are the Dynkin labels of  $R$  and  $m_i$  the co-marks of  $G$ . The values for  $\tilde{h}$  and  $m_i$  can be found in Table 1 of Reference [13].

Applying Eq. (2) to the heterotic string ( $c \leq 22$ ) one finds for the exceptional groups  $k \leq 4, 3, 2$  for  $E_6, E_7, E_8$ , respectively. Eq. (3) then reveals that only the fundamental and adjoint representations of these groups can appear in the *massless* spectrum ( $h \leq 1$ ). Hence, the **351** and **351'** which are often used in  $E_6$  model building are not permitted.

$SO(4N+2)$  for  $N \geq 2$  are candidate gauge groups for unified model building, with each fermion generation in the non-selfconjugate, anomaly-free, basic spinor representations  $4^N$ . However, Eq. (2) shows that for  $N > 3$  the level  $k$  cannot be greater than 2. On the other hand, we find for the basic spinor representation of  $SO(2N)$ ,

$$\text{at level 1} \quad h_{2N-1} = \frac{N}{8}, \quad (6)$$

and

$$\text{at level 2} \quad h_{2N-1} = \frac{2N-1}{16}, \quad (7)$$

which both exceed 1 for  $N \geq 9$ , so that  $SO(18)$  and bigger orthogonal groups are ruled out from heterotic string model building. For  $SO(10)$  and  $SO(14)$  we find  $k \leq 7$  and  $k \leq 3$ , respectively. Eq. (5) can now be used to establish that primary fields transforming in the **120** or **126** of  $SO(10)$  can appear<sup>5</sup> for  $k \geq 2$ . They play the analogous roles for the discussion of mass matrices like the **351** and **351'** of  $E_6$ , respectively. However, as already noted in [17], condition (3) is often much stronger. Indeed, the **120** can only appear in the massless spectrum for  $k \geq 3$ , and the **126** (whose Clebsch-Gordon coefficients could account for  $m_s - m_\mu$  unification) for  $k \geq 5$ .

As for  $SU(N)$  GUT models from the heterotic string, Eq. (2) yields  $N \leq 12$  for level  $k = 2$ . In general, no further restrictions arise, since the conformal dimension of the  $M$ -index complete antisymmetric representation of  $SU(N)$  is at level 1 given by

$$h = \frac{M(N-M)}{2N} \quad (< 1 \text{ for } M = 1, 2). \quad (8)$$

Thus, the two-index antisymmetric and fundamental representations as used, for example, for minimal  $SU(5)$  are allowed to appear in the massless spectrum for any  $N$ . The **45** of  $SU(5)$ , used to achieve more realistic fermion mass matrices, is allowed by both Eq. (3) and Eq. (5) to be massless for  $k \geq 2$ . The level two representations **50** and **75**, employed in the

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<sup>5</sup>At level 2, all two-index (vector, spinor or mixed) representations, as well as arbitrary-index complete antisymmetric representations are allowed.

missing partner mechanism [20], can appear in the massless spectrum for  $k \geq 4$  and  $k \geq 3$ , respectively. Note, however, that for this mechanism explicit heavy mass terms are needed to keep the unwelcome states inside the **50** and **75** at the GUT scale. Therefore, it seems preferable to construct models where these representations have a mass which would be a fraction of the Planck mass, using explicitly the fact that the unification and string scales are close to each other. For example, one could make use of the small mass increments in  $Z_N$ -orbifolds of high twist order  $N$ . There are two more advantages for doing so: if the extra **50** and **75** representations are too close to the GUT scale, then the gauge coupling would become strong below the Planck scale; on the other hand, if they are too close to the Planck scale the see-saw type triplet mass would be significantly below the GUT scale, which would lead to too rapid proton decay [21].

In fact, if one is interested in massive states, it is important to note that Eq. (5) is a restriction on the unitary highest weight representations of the conformal field theory, i.e. it constraints the primary fields. However, given any primary field  $\phi_R$ , through secondary fields one will find all representations of  $G$  in the same congruency class as  $R$  at some mass level. By the same token, the Kac-Moody currents being themselves descendents of the identity field can give rise to massless adjoint representations already at level one. Examples are gauge bosons, gauginos, as well as adjoint scalars in  $N = 2$  supermultiplets. However, these massless adjoints are always non-chiral [13].

### 3 Asymmetric Orbifolds

In this section, I will describe how one can use the asymmetric orbifold construction to manifestly arrive at level two models. I will not consider the possibility of quantized Wilson lines the inclusion of which is straightforward, but more tedious in practice. When there are no Wilson lines, the internal space part decouples from the gauge part and the Narain vectors [22] are simply given by

$$P_{L/R} = \frac{m}{2} - bn \pm gn, \quad (9)$$

where  $n$  and  $m$  are integer valued  $d$  dimensional winding and momentum vectors. The metric  $g$  is normalized (using  $\alpha'$ ) such that the Narain scalar product is given by

$$P^2 \equiv P_L^2 - P_R^2 \equiv P_L g^{-1} P_L - P_R g^{-1} P_R. \quad (10)$$

With this convention, at a point of enhanced symmetry,  $g$  is one quarter of the respective Cartan metric and the tensor  $b$  is any antisymmetric counterpart of  $g$ .

An orbifold twist  $\theta$  must leave the conformal dimensions  $P_L^2/2$  and  $P_R^2/2$  invariant. Hence, any twist has the form

$$P_{L/R} \rightarrow P'_{L/R} = \theta_{L/R}^* P_{L/R}, \quad (11)$$

with  $\theta_{L/R}^* \equiv \theta_{L/R}^{T^{-1}}$ , and the two conditions,

$$\theta_{L/R}^t g \theta_{L/R} = g, \quad (12)$$

must be satisfied. The transformed winding and momentum vectors are then straightforwardly obtained, and given by

$$\begin{aligned} n' &= a_{nn}n + a_{nm}m, \\ m' &= a_{mn}n + a_{mm}m, \end{aligned} \quad (13)$$

where

$$\begin{aligned} a_{nn} &= \frac{1}{2}[(\theta_L + \theta_R) - (\theta_L - \theta_R)g^{-1}b], \\ a_{nm} &= \frac{1}{4}(\theta_L - \theta_R)g^{-1}, \\ a_{mn} &= \tilde{\theta} + (\theta_L^* - \theta_R^*)g - b(\theta_L - \theta_R)g^{-1}b, \\ a_{mm} &= \frac{1}{2}[(\theta_L^* + \theta_R^*) + b(\theta_L - \theta_R)g^{-1}], \end{aligned} \quad (14)$$

and

$$\tilde{\theta} = b(\theta_L + \theta_R) - (\theta_L^* + \theta_R^*)b. \quad (15)$$

The blocks  $a_{ij}$  fill up a  $2d \times 2d$  dimensional integer matrix,

$$\Theta = \begin{pmatrix} a_{nn} & a_{nm} \\ a_{mn} & a_{mm} \end{pmatrix}, \quad (16)$$

which is the twist matrix acting in an Euclidean lattice with metric

$$G = \begin{pmatrix} 2(g-b)g^{-1}(g+b) & bg^{-1} \\ -g^{-1}b & \frac{1}{2g} \end{pmatrix}. \quad (17)$$

Note, that for symmetric twists,  $\theta \equiv \theta_L \equiv \theta_R$ ,  $n$  transforms homogeneously,

$$n \rightarrow \theta n, \quad (18)$$

whereas

$$m \rightarrow \theta^* m + \tilde{\theta} n \quad (19)$$

receives a winding admixture whenever  $2[b\theta - \theta^*b] \neq 0$ . One sees that  $\theta$  and  $\tilde{\theta}$  are integer valued matrices and that  $b$  can assume quantized values similar to Wilson lines. The  $T$ -dual twist [23],

$$\hat{\theta}_{L,R} = \theta - 2(b \mp g)\tilde{\theta} = (g \mp b)\theta^* \frac{1}{g \mp b}, \quad (20)$$

is, however, asymmetric. Duality is here not a symmetry in moduli space, but relates symmetric and asymmetric orbifolds. That symmetric and asymmetric orbifolds are closely related is also evident from the fact that the K3 surface has both symmetric and asymmetric orbifold points [24]. Thus, an asymmetric orbifold can have a very clear geometrical interpretation.

I will now discuss a simple asymmetric orbifold at level 2. Although it is realized in  $D = 6$  uncompactified dimensions with simple supersymmetry, it is related to the non-supersymmetric  $D = 10$  heterotic string theory with gauge group  $E_8^{k=2}$  [25]. It will (sometimes with modifications of the compactification lattice) play the role of the untwisted sector of the  $D = 4$  models to be discussed later.

It is a  $Z_2$  orbifold in which the four dimensional internal space part on the bosonic side of the heterotic side remains untouched,  $\theta_L = \mathbf{1}$ , while  $\theta_R = -\mathbf{1}$ , and the two  $E_8$  factors

are interchanged. The compactification lattice can be uniquely determined by looking at the degeneracy in the twisted sector. In an even self-dual lattice the quantity [18]

$$D = \sqrt{\frac{N_L^f \times N_R^f}{\det g_{inv}}}, \quad (21)$$

where  $N_{L/R}^f$  are the numbers of left and right fixed points, is always an integer.  $g_{inv}$  is the metric of the invariant Narain sublattice. In the case at hand, the number of left fixed points (from the gauge part) is  $2^8$ . From the metric of the invariant gauge lattice (the *diagonal*  $E_8$ ) we find

$$\det 2g_{E_8} = 2^8, \quad (22)$$

as well, because the invariant  $E_8$  vectors have double length squares. Thus the degeneracy from the gauge part is  $D_{gauge} = 1$ . The number of right fixed points is  $2^4$ . Thus, the metric of the invariant space lattice must have determinant 1, 4 or 16. It is given by  $4g$  which we want to be the Cartan matrix of a simply-laced Lie algebra. A look at the semi-simple Lie algebras with rank 4 reduces the choice to  $SU(2)^4$  or  $SO(8)$ . Both give rise to integral twists  $\Theta$ . However, the  $SU(2)^4$  lattice must be rejected, as it will become clear later that this lattice does not satisfy the level matching condition. On the other hand the  $SO(8)$  lattice is adequate ( $\det g_{SO(8)} = 4$ ,  $D = 2$ ).

The massless states in the untwisted sector are easily obtained: the supergravity and dilaton multiplets in six dimensions; the super Yang-Mills multiplets from the diagonal  $E_8$  and the enhanced  $SO(8)$ ; and one hypermultiplet transforming in the adjoint of  $E_8$ . As for the twisted sector, we have already seen that  $D = 2$ , but since there are only 2 spinors per fixed point and a hypermultiplet contains 4 fermionic states, we find an effective degeneracy  $D_{eff} = 1$ . The vacuum energy for the left movers is  $1/2$ . Upon world sheet modular transformations we have the lattice with metric  $g_{inv}^{-1}$  at our disposal. Due to the self-duality of the  $E_8$  lattice, the inverse of  $2g_{E_8}$  is up to an integral similarity transformation simply given by  $\frac{1}{2}g_{E_8}$ . Thus, for massless states we have to look for solutions of

$$P_{E_8}^2/4 = 1/2, \quad (23)$$

which are just the roots. Combined with the 8 half-oscillator states they give rise to a twisted adjoint representation of  $E_8$ . Finally, for states with  $P_{E_8}^2 = 0$ , we must consider the lattice with metric  $g_{SO(8)}^{-1}$ , the weight space of  $SO(8)$ . Massless states are the ones with length squares equal to unity and correspond to the triality symmetric combination  $\mathbf{8}_v + \mathbf{8}_s + \mathbf{8}_c$ .

At this point level matching would break down, had we chosen the  $SU(2)^4$  lattice. The states in the  $SU(2)$  weight lattices have length squares corresponding to conformal dimensions  $k^2/4$ , with  $k \in \mathbf{Z}$ . Hence, states corresponding to odd  $k$  do not match with states from the right hand side which are half integer spaced.

In total we have matter transforming under  $E_8 \times SO(8)$  like

$$2(\mathbf{248}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}_v + \mathbf{8}_s + \mathbf{8}_c).$$

Note that

$$N_H - N_V = 244, \quad (24)$$

where  $N_H$  and  $N_V$  are the numbers of hypermultiplets and vector multiplets, respectively, as is required in six dimensional supergravity with precisely one tensor multiplet (the dilaton multiplet) for the cancellation of gravitational anomalies. Also, as usual in  $D = 6$ , cancellation of gauge and mixed gauge/gravitational anomalies constitutes a highly non-trivial check. The anomaly has to factorize so that it can be canceled by the Green-Schwarz mechanism [26]. Here it does, and there is a new feature at higher level. The anomaly polynomial is given by

$$i(2\pi)^3 I = -\frac{1}{16}[\text{tr} R^2 - \frac{1}{6}\text{Tr} F_{SO(8)}^2 - \frac{1}{15}\text{Tr} F_{E_8}^2] \times [\text{tr} R^2 + \frac{1}{4}\text{Tr} F_{SO(8)}^2 - \frac{1}{10}\text{Tr} F_{E_8}^2], \quad (25)$$

where traces in the adjoint representations of the gauge groups are used. Note, that then the coefficients in the first factor are simply given by  $k/\hbar$ . One can use this fact to show that in  $D = 6$  there can only be three possibilities for  $E_8$ : (1) no adjoint representation corresponding to  $k = 1$  like in the case of compactifying the heterotic string on  $K_3$ ; (2) one adjoint corresponding to  $k = 0$ , i.e. the **248** must be part of an  $N = 2$  gauge multiplet; and (3) two adjoints as in the  $k = 2$  case just discussed. A larger number of **248** representations, would lead to irrational coefficients in the anomaly polynomial.

For compactification to  $D = 4$ ,  $N = 2$ , the above model can now be used by either attaching a torus,  $T_2$ , or by changing the lattice to e.g. the  $SO(12)$  or the  $E_6$  lattice. The breaking to  $N = 1$  goes along with a simultaneous breaking of  $E_8$ . One can act in each  $E_8$  with a twist or shift of order  $N$ , but it must be the same action in both  $E_8$  factors. This defines an auxiliary  $Z_N$  orbifold in its own right which would give rise to a gauge group  $G_8 \times G_8 \times G_6$ .  $G_8$  is the gauge group which (upon permutation of the two sets of  $E_8$  gauge coordinates) will be promoted to level 2 and  $G_6$  is the enhanced gauge group arising from the space part.

To summarize, the class of models described in detail below, are  $Z_2 \times Z_N$  orbifolds, where  $Z_2$  refers to the level 2,  $N = 2$  models above, and  $Z_N$  breaks one supersymmetry and  $E_8$  to  $G_8$ . In the simplest case of a  $Z_2 \times Z_2$  orbifold one can obtain only the non-chiral groups  $G_8 = E_7 \times SU(2)$  and  $SO(16)$ . On the other hand,  $Z_2 \times Z_3$  and  $Z_2 \times Z_4$  orbifolds yield many chiral possibilities as shown in Table 1, and  $Z_2 \times Z_6$  orbifolds include  $SU(5) \times SU(4) \times U(1)$ . In the next section, I will exploit the most interesting  $Z_2 \times Z_3$  case, namely  $E_6 \times SU(3)$  models. Section 5 focuses on  $Z_2 \times Z_4$  orbifolds with gauge groups  $SO(10) \times SU(4)$  and  $E_6 \times SU(2) \times U(1)$ .

$m$	$Z_2 \times Z_2$	$Z_2 \times Z_3$	$Z_2 \times Z_4$
1	$E_7 \times SU(2)$	$E_7 \times U(1)$	$E_7 \times U(1)$
2	$SO(16)$	$SO(14) \times U(1)$	$SO(14) \times U(1)$
3		$E_6 \times SU(3)$	$E_6 \times SU(2) \times U(1)$
4		$SU(9)$	$SU(8) \times U(1)$
5			$SO(12) \times SU(2) \times U(1)$
6			$SO(10) \times SU(4)$
7			$SU(7) \times SU(2) \times U(1)$

Table 1: Possible groups  $G_8$  at level  $k = 2$  from  $Z_2 \times Z_N$  orbifolds. The groups are listed according to the twist order  $N$  and an integer  $m$ . The gauge contribution to the vacuum energy of the first twisted sector is given by  $E_{vac}^{gauge} = 2m/N^2$ .

## 4 $Z_2 \times Z_3$ orbifolds with $E_6$ gauge group at level 2

The  $Z_3$  action of the  $Z_2 \times Z_3$  orbifold in the gauge part is the same as in the case of the standard “ $Z$  manifold” [27], only that here both  $E_8$  factors are twisted<sup>6</sup>. It is a peculiarity of prime orbifolds, that they lead to modular invariant partition functions when one uses standard embeddings without twisting the left space part. That opens up the two possibilities of twisting all left internal coordinates by a  $Z_3$  rotation, or alternatively, leaving all of them untouched.

On the right hand side we have two choices of supersymmetric  $Z_3$  twists:  $Z_3$  could act like in the case of the  $Z$  manifold by rotating all three pairs of right handed internal coordinates, or it could rotate just two pairs, in which case it leads by itself to  $N = 2$  supersymmetry and I will refer to it as  $Z'_3$ . Let the  $Z_2$  action take place in the first two complex coordinates of the right hand side and the  $Z'_3$  action in the last two. Then,  $Z_2 \times Z_3$  corresponds to the usual  $Z_6$  orbifold and  $Z_2 \times Z'_3$  to  $Z'_6$ . In summary, we have four possibilities to choose the internal *twist eigenvalue structure*:

$$\begin{aligned} \text{A} & \left( \begin{array}{cc} Z_3 & , & Z_6 \end{array} \right), \\ \text{B} & \left( \begin{array}{cc} \mathbf{1} & , & Z_6 \end{array} \right), \\ \text{C} & \left( \begin{array}{cc} Z_3 & , & Z'_6 \end{array} \right), \\ \text{D} & \left( \begin{array}{cc} \mathbf{1} & , & Z'_6 \end{array} \right). \end{aligned} \tag{26}$$

The next step consists of specification of the lattices.

Cases A through C: In these cases there is at least one  $Z_3$  involved (as opposed to  $Z'_3$ ). That means one has to look for groups possessing an  $SU(3)^3$  subgroup. The two possibilities are  $E_6$  and  $SU(3)^3$  itself. However, the  $SU(3)^3$  root lattice must be rejected, because a  $Z_6$  twist cannot act asymmetrically in an  $SU(3)$  lattice.

On the other hand, a consistent twist acting in the  $E_6$  lattice can be constructed. It is convenient to use the  $SU(3)^3$  basis of  $E_6$ . Define the simple roots of one  $SU(3)$  by

$$e_1 = (\sqrt{2}, 0), \quad e_2 = \left(-\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right). \tag{27}$$

The Cartan metric is

$$g_{SU(3)} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{28}$$

and the fundamental weights are given by

$$\tilde{e}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right), \quad \tilde{e}_2 = \left(0, \sqrt{\frac{2}{3}}\right). \tag{29}$$

Note, that

$$\begin{aligned} e_1 + e_2 &= \tilde{e}_1 + \tilde{e}_2, \\ \tilde{e}_2 - \tilde{e}_1 &= \tilde{e}_1 - e_1, \end{aligned} \tag{30}$$

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<sup>6</sup>In Reference [28] this is called symmetric embedding.

which is useful for constructing the twists. Distinguish between the  $SU(3)$  factors by unprimed, primed and doubly primed symbols. Then, a basis of  $E_6$  is given by

$$\begin{aligned} f_1 &= (e_1, 0, 0), & f_2 &= (e_2, 0, 0), \\ f_3 &= (0, e'_1, 0), & f_4 &= (0, e'_2, 0), \\ f_5 &= (\tilde{e}_1, \tilde{e}'_1, \tilde{e}''_1), & f_6 &= (\tilde{e}_2, \tilde{e}'_2, \tilde{e}''_2), \end{aligned} \quad (31)$$

corresponding to the metric

$$g_{E_6} = \begin{pmatrix} g_{SU(3)} & 0 & \mathbf{1} \\ 0 & g_{SU(3)} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 3g_{SU(3)}^{-1} \end{pmatrix}, \quad (32)$$

with inverse

$$g_{E_6}^{-1} = \begin{pmatrix} 2g_{SU(3)}^{-1} & g_{SU(3)}^{-1} & -\mathbf{1} \\ g_{SU(3)}^{-1} & 2g_{SU(3)}^{-1} & -\mathbf{1} \\ -\mathbf{1} & -\mathbf{1} & g_{SU(3)} \end{pmatrix}. \quad (33)$$

If we denote a  $Z_3$  twist in one  $SU(3)$  by

$$\theta = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad (34)$$

we can write for the six dimensional  $Z_3$  twist matrix

$$\theta_3 = \text{diag}(\theta, \theta, \theta^*). \quad (35)$$

If we further define

$$\Delta \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (36)$$

we have for  $Z'_3$

$$\theta'_3 = \begin{pmatrix} \theta & 0 & -\Delta \\ 0 & \theta & -\Delta \\ 0 & 0 & \mathbf{1} \end{pmatrix}. \quad (37)$$

Finally, we have to specify in which way the  $Z_2$  acts in our lattice. We choose it in such a way that it permutes the first two  $SU(3)$  factors in addition to negating all vectors (in order to get the correct number of eigenvalues  $-1$ ),

$$\theta_2 = \begin{pmatrix} 0 & -\mathbf{1} & 0 \\ -\mathbf{1} & 0 & 0 \\ 0 & 0 & -\mathbf{1} \end{pmatrix}. \quad (38)$$

It can be checked that  $g_{E_6}$  (in the sense of Eq. (12)),  $\theta_3$ ,  $\theta'_3$  and  $\theta_2$  all mutually commute. Now we can simply define  $\theta_6 = \theta_2\theta_3$  and  $\theta'_6 = \theta_2\theta'_3$ . For the antisymmetric tensor  $b$  we choose

$$b_{E_6} = \begin{pmatrix} \sigma & 0 & \mathbf{1} \\ 0 & \sigma & \mathbf{1} \\ -\mathbf{1} & -\mathbf{1} & \sigma \end{pmatrix}, \quad (39)$$

with

$$\sigma \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (40)$$

but the distribution of signs plays no role. One still has to show that the twists defined this way lead to an integer matrix  $\Theta$  when inserted into the expressions (14). This turns out to be true for all cases A through D.

Case D does not involve the  $Z_3$  (only  $Z'_3$ ) twist, and one can try to find more lattices than just the root lattice of  $E_6$ . Indeed, since case D possesses 12 right fixed points, the metric of the invariant sublattice could have a determinant of either 3 or 12. The  $E_6$ -lattice discussed before corresponds to the former case since  $\det g_{E_6} = 3$ . A lattice with determinant 12 is the root lattice of  $SO(8) \times SU(3)$ . We define  $Z'_3$  such that it acts in the explicit  $SU(3)$  factor and in an  $SU(3)$  subgroup of  $SO(8)$ . The  $Z_2$  acts by negating the  $SO(8)$  roots. Again, the integer condition following from (14) can be checked to be satisfied.

I will refer to the orbifolds defined by the eigenvalue structures A through D acting in the  $E_6$  lattice as models A through D and model E will be the one realized in  $SO(8) \times SU(3)$ . The resulting spectra are displayed in Table 2. In the following, I discuss in some detail the spectrum calculation for model A. In the second and third twisted sector, we will find the phase ambiguities alluded to in the introduction. In all models A through E the phases can be fixed by requiring CPT invariance and cancellation of anomalies. Clearly, this is not a satisfactory state of affairs, and in section 5, I will introduce a systematic way to compute such phases.

## Untwisted sector

For the NSR-fermions I will use the shift description, i.e. I use bosonized world sheet fermions and act with shift vectors in the vector and spinor congruency classes of the  $SO(8)$  lattice. In the explicit discussion, I restrict myself to positive helicity spinor states (last entry =  $+1/2$ ), as the remaining states are just the CPT and supersymmetry partners. The relevant shift vectors  $v$  corresponding to the  $Z_6$  and  $Z'_6$  twists discussed before will be taken to be

$$\begin{aligned} v_6 &= (+\tfrac{1}{6}, +\tfrac{1}{6}, -\tfrac{1}{3}, 0), \\ v'_6 &= (+\tfrac{1}{6}, +\tfrac{1}{3}, -\tfrac{1}{2}, 0), \\ v_3 &= (+\tfrac{1}{3}, +\tfrac{1}{3}, -\tfrac{2}{3}, 0), \end{aligned} \quad (41)$$

where for comparison the shift vector for the standard  $Z_3$  orbifold is also shown. The positive helicity ground states  $h$  with their shift phases  $e^{2\pi i h v}$  for the three cases are ( $\alpha = e^{2\pi i/6}$ )

$\theta_R$	$Z_6$	$Z'_6$	$Z_3$
$(+\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2})$	1	1	1
$(+\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, +\tfrac{1}{2})$	$\alpha$	$\alpha$	$\alpha^2$
$(-\tfrac{1}{2}, +\tfrac{1}{2}, -\tfrac{1}{2}, +\tfrac{1}{2})$	$\alpha$	$\alpha^2$	$\alpha^2$
$(-\tfrac{1}{2}, -\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2})$	$\alpha^4$	-1	$\alpha^2$

(42)

	A	B	C	D	E
$G_6$	$SU(3)^3$	$E_6$	$SU(3)^3$	$E_6$	$SU(3) \times SO(8)$
U	$3(\mathbf{27}, \bar{\mathbf{3}}, 1, 1, 1)$ $(1, 1, \mathbf{3}, \mathbf{3}, \mathbf{3})$	$3(\mathbf{27}, \bar{\mathbf{3}}, 1)$	$(\mathbf{78}, 1, 1, 1, 1)$ $(1, \mathbf{8}, 1, 1, 1)$ $(\mathbf{27}, \mathbf{3}, 1, 1, 1)$ $(\mathbf{27}, \bar{\mathbf{3}}, 1, 1, 1)$ $(1, 1, \bar{\mathbf{3}}, \bar{\mathbf{3}}, \bar{\mathbf{3}})$	$(\mathbf{78}, 1, 1)$ $(1, \mathbf{8}, 1)$ $(\mathbf{27}, \mathbf{3}, 1)$ $(\mathbf{27}, \bar{\mathbf{3}}, 1)$	$(\mathbf{78}, 1, 1, 1)$ $(1, \mathbf{8}, 1, 1)$ $(\mathbf{27}, \mathbf{3}, 1, 1)$ $(\mathbf{27}, \bar{\mathbf{3}}, 1, 1)$
T1	$(1, \mathbf{3}, \mathbf{3}, \mathbf{1}, \mathbf{1})$	$(\mathbf{27}, \bar{\mathbf{3}}, 1)$	$2(1, \mathbf{3}, \mathbf{3}, \mathbf{1}, \mathbf{1})$	$2(\mathbf{27}, \bar{\mathbf{3}}, 1)$	$(\mathbf{27}, \bar{\mathbf{3}}, 1, 1)$ $(1, \mathbf{3}, \mathbf{3} + \bar{\mathbf{3}}, 1)$
T2	$(1, \mathbf{6}, \bar{\mathbf{3}}, 1, 1)$ $2(1, \bar{\mathbf{3}}, \bar{\mathbf{3}}, 1, 1)$	$3(\mathbf{27}, \mathbf{3}, 1)$	$(1, \bar{\mathbf{6}}, \mathbf{3}, 1, 1)$ $(1, \bar{\mathbf{3}}, \bar{\mathbf{3}}, 1, 1)$	$(\mathbf{27}, \mathbf{3}, 1)$ $(\mathbf{27}, \bar{\mathbf{3}}, 1)$	$(\mathbf{27}, \bar{\mathbf{3}}, 1, 1)$ $(\mathbf{27}, \mathbf{3}, 1, 1)$ $(1, \mathbf{6}, \mathbf{3} + \bar{\mathbf{3}}, 1)$ $(1, \mathbf{3}, \mathbf{3} + \bar{\mathbf{3}}, 1)$
T3	$(\mathbf{78}, 1, 1, 1, 1)$ $(1, \mathbf{8}, 1, 1, 1)$ $(\mathbf{27}, \mathbf{3}, 1, 1, 1)$	$(\mathbf{78}, 1, 1)$ $(1, \mathbf{8}, 1)$ $(\mathbf{27}, \mathbf{3}, 1)$	$2(\mathbf{27}, \mathbf{3}, 1, 1, 1)$	$2(\mathbf{27}, \mathbf{3}, 1)$	$(\mathbf{78}, 1, 1, 1)$ $(1, \mathbf{8}, 1, 1)$  $(\mathbf{27}, \bar{\mathbf{3}}, 1, 1)$ $(1, 1, 1, \mathbf{8}_v + \mathbf{8}_s + \mathbf{8}_c)$

Table 2: Models from asymmetric  $Z_2 \times Z_3$  orbifold with gauge group  $[E_6 \times SU(3)]^{k=2} \times G_6^{k=1}$ . U denotes the untwisted sector, while T1, T2 and T3 are the twisted sectors.

The gauge shift vector is defined as

$$V_3^{gauge} = (v_3, 0^4; v_3, 0^4). \quad (43)$$

Consider the **128** spinor representation of  $SO(16) \subset E_8$ , which decomposes into 8 groups of **16** ( $\bar{\mathbf{16}}$ ) of  $SO(10)$ . Labeling these groups by their first three entries one finds the following twist phases and gauge transformation properties under  $E_6 \times SU(3)$ :

	$S$	$A$	
$(+\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2})$	1	-1	$(\mathbf{78}, 1) + (1, \mathbf{8})$
$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	1	-1	
$(+\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	$\alpha^2$	$\alpha^5$	$(\mathbf{27}, \mathbf{3})$
$(-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2})$	$\alpha^2$	$\alpha^5$	
$(-\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2})$	$\alpha^2$	$\alpha^5$	
$(-\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2})$	$\alpha^4$	$\alpha$	$(\mathbf{27}, \bar{\mathbf{3}})$
$(+\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2})$	$\alpha^4$	$\alpha$	
$(+\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2})$	$\alpha^4$	$\alpha$	

(44)

In this table  $S$  and  $A$  refer to the symmetric and antisymmetric linear combinations of the two  $E_8 \times E_8$  vectors. The twist invariant states will yield the gauge bosons. We can see that we have untwisted adjoint matter, iff there is a helicity vector with twist phase  $-1$ . This is not the case for model A where we find fields transforming like

$$3(\mathbf{27}, \bar{\mathbf{3}}, \mathbf{1}),$$

under the unbroken gauge group  $[E_6 \times SU(3)]^{k=2} \times G_6^{k=1}$ . For model A,  $G_6 = SU(3)^3$  since only 24 orbits of the 72 roots of  $E_6$  are twist invariant. 48 orbits and the 6 oscillators transform under the twist. Therefore there are additional untwisted matter fields transforming like

$$(\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{3}, \mathbf{3}).$$

In this class of models untwisted adjoint matter appears precisely when the untwisted sector is non-chiral.

For comparison I have chosen a convention in which the ordinary  $Z_3$  orbifold at the point of maximally enhanced gauge symmetry has the spectrum

$$\begin{aligned} 3(\mathbf{1}, \mathbf{27}, \underline{\mathbf{3}}, \underline{\mathbf{1}}, \underline{\mathbf{1}}, \underline{\mathbf{1}}), \\ 3(\mathbf{1}, \mathbf{1}, \underline{\bar{\mathbf{3}}}, \underline{\bar{\mathbf{3}}}, \underline{\bar{\mathbf{3}}}, \underline{\mathbf{1}}), \end{aligned}$$

under  $E_8 \times E_6 \times SU(3)^4$ , where underlining means to take all permutations.

### First twisted sector

There is only one massless spinor in this sector, namely

$$p = \left(-\frac{1}{3}, -\frac{1}{3}, +\frac{1}{6}, +\frac{1}{2}\right), \quad (45)$$

having positive helicity. The degeneracy from the space part is  $D = 9$ , and the corresponding fixed points are charged under the enhanced gauge group. The degeneracy from the gauge part is 1 since the number of gauge fixed points cancels the volume factor of the invariant gauge lattice. The latter is the *diagonal*  $E_8$ , with a shift vector  $\tilde{V}_3^{gauge}$  acting in it. It is important to realize that  $\tilde{V}_3^{gauge}$  is given by *twice*  $V_3^{gauge}$  truncated to one  $E_8$ ,

$$\tilde{V}_3^{gauge} = 2(v_3, 0^4). \quad (46)$$

The vacuum energy from the gauge (space) part is  $1/2$  ( $1/3$ ) so that we have to look for states satisfying

$$\frac{(P_{E_8} + \tilde{V}_3^{gauge})^2}{4} = \frac{1}{6}, \quad (47)$$

corresponding to a  $(\mathbf{1}, \mathbf{3})$ . In order to determine the transformation of the 9 fixed points, it suffices to compare with the spectrum of the  $Z_3$  orbifold. In its simply twisted sector the  $\mathbf{27}$  come together with triplets of  $SU(3)$  and since the helicities are positive in either case, we have matter transforming like

$$(\mathbf{1}, \mathbf{3}, \underline{\mathbf{3}}, \underline{\mathbf{1}}, \underline{\mathbf{1}}).$$

## Second twisted sector

The degeneracy factor for model A is before projecting onto  $Z_2$  invariant states easily seen to be given by  $D = 27$ . However, in the case of non-trivial invariant lattices, it may be less straightforward to find the degeneracy factor, and I will now shortly describe how to find it for model E. One notes first that from the  $SU(3)$  factor we have  $D = 1$  since the contribution of the three right-chiral fixed points is canceled by the invariant left-chiral  $SU(3)$  root lattice. We have one more  $Z_3$  acting in a subgroup of  $SO(8)$ , the rest being unrotated. We have to find the determinant of this invariant Narain sublattice. Combining the left and right parts of the Narain lattice, one finds an  $SO(8) \times SO(8)$  sublattice. The metric of this lattice has determinant 16, while a self-dual lattice must have determinant one. There must be extra states having entries simultaneously in the left and the right part of the Narain lattice. Integrality of self-dual lattices requires them to be weight vectors. Furthermore, if they correspond to Kaluza-Klein states, they are left-right symmetric. Thus, one has to include the congruency class  $(\mathbf{8}_v, \mathbf{8}_v)$  which after passing to a Euclidean metric enlarges  $SO(8) \times SO(8)$  to  $SO(16)$ . Similarly, one has to add the classes  $(\mathbf{8}_s, \mathbf{8}_s)$  and  $(\mathbf{8}_c, \mathbf{8}_c)$ , which combined transform as a  $\mathbf{128}$  of  $SO(16)$  so that one finally arrives at the  $Spin(16)/Z_2 \equiv E_8$  lattice. In other words, for a compactification on a lattice with metric  $g = 1/4g_{SO(8)}$  and where  $b$  is its antisymmetric counterpart, one finds from Eq. (17),  $G = g_{E_8}$ . Now a  $Z_3$  twist acting in an  $SU(3)$  subgroup of  $E_8$  leaves an  $E_6$  root lattice invariant, the metric of which has determinant 3, cancelling the contribution from the three fixed points and there is an overall degeneracy of  $D = 1$ .

Similarly, the Narain lattice of model D contains the congruency classes  $(\mathbf{78}, \mathbf{1}) + (\mathbf{1}, \mathbf{78}) + (\mathbf{27}, \mathbf{27}) + (\overline{\mathbf{27}}, \overline{\mathbf{27}})$  of  $E_6 \times E_6$ . The  $Z_3$  acts in an  $SU(3) \times SU(3)$  subgroup of the right-moving  $E_6$ , leaving fixed the root lattice of  $E_6 \times SU(3)$ . Again, the 9 right fixed points cancel against its volume factor,  $\det g_{E_6} \det g_{SU(3)} = 9$ , yielding a degeneracy of  $D = 1$ . These results can be checked explicitly, by solving the equation  $\Theta N = N$ , with  $\Theta$  from Eq. (16) and  $N^T = (n^T, m^T)$ .

The massless spinor in this sector is given by

$$p = \left(-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, +\frac{1}{2}\right). \quad (48)$$

As for the gauge part, for model A we have to consider solutions to

$$\frac{(P_{E_8} + 2V_3^{gauge})^2}{2} = \frac{2}{3}, \quad (49)$$

which correspond to

$$(\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{3}). \quad (50)$$

Now one has to project onto  $Z_2$  invariant states. As mentioned there are 27 fixed points under  $Z_3$ . Four complex *chiral* dimensions are purely  $Z_3$  rotated so that there is no  $Z_2$  phase, but two complex dimensions behave non-trivially. For each such complex dimension the origin is fixed and there is the symmetric and the antisymmetric combination of the remaining  $Z_3$  fixed points. Hence, we have two invariant (symmetric) and one antisymmetric combination, and I will denote this by  $2^s + 1^a$ . Combining everything yields

$$D = \sqrt{3^4(2^s + 1^a)^2} = \sqrt{3^4(1^s + 2^a)^2} = 9(2^s + 1^a) \text{ or } 9(1^s + 2^a), \quad (51)$$

and one finds an ambiguity due to the presence of the square root. Unfortunately, it is impossible to resolve this ambiguity in this framework. In section 5, I will introduce a different, yet equivalent method to describe these sort of models. It will use a shift description also for the internal space part, so that all phases can be fixed unambiguously. Of course, one may also fix the phases by requiring the model to be free of gauge anomalies. In any case it turns out that the correct choice is the latter option in Eq. (51) and to take the symmetric combination of (50),

$$\mathbf{3} \times \mathbf{3} = \mathbf{6}^s + \mathbf{\bar{3}}^a, \quad (52)$$

with multiplicity 9 and the antisymmetric one with multiplicity 18. These states transform in addition under the enhanced  $SU(3)^3$ . We conclude there is matter transforming like

$$(\mathbf{1}, \mathbf{6}, \mathbf{\bar{3}}, \mathbf{1}, \mathbf{1}),$$

$$2(\mathbf{1}, \mathbf{\bar{3}}, \mathbf{\bar{3}}, \mathbf{1}, \mathbf{1}).$$

A similar ambiguity appears in model C. Here are two opposite helicities  $p'_\pm$  transforming with a relative sign under  $Z_2$ ,  $e^{6\pi i p'_\pm v'_6} = \alpha^5, \alpha^2$ . Obviously, there must be extra overall phases combining to  $\pm\alpha$  in order to reduce above phases to  $\pm 1$ . Again we cannot determine them within in the present framework. Although it is clear that one has to take both the symmetric and antisymmetric combinations appearing in the product (52), the overall phase is important to determine chiralities. Here it turns out that one has to take the antisymmetric combination for the positive helicity states (yielding antitriplets) and the symmetric combination for negative helicity states (producing antisextets). A similar situation as in model C occurs also in model E, but models B and D happen to be free of any phase ambiguities.

### Third twisted sector

In this sector there are two massless spinors with opposite helicities,

$$p_\pm = (0, 0 \mp \frac{1}{2}, \pm \frac{1}{2}). \quad (53)$$

The degeneracies for all models is  $D = 2$ , as was the case for the  $N = 2$  model discussed at the end of section 3. This is evident for model E, but the fact that the asymmetric  $Z_2$  action in the  $E_6$  lattice indeed yields an invariant lattice of determinant 4 must be checked explicitly. From the gauge part one has the solutions of Eq. (23) plus the eight half-integer oscillators at ones disposal.

Now one has to project onto  $Z_3$  invariant states. One obtains for the positive (negative) helicity vector  $e^{4\pi i p_\pm v_6} = \alpha^2 (\alpha^4)$ . We are looking for  $Z_2$  fixed points which are not fixed under  $Z_3$ . Consider first the complex dimension being  $Z_6$  twisted. There is the twist invariant origin and the three non-trivial  $Z_2$  fixed points transform as a triplet under  $Z_3$ . Hence, in a notation similar to the one of the previous sector one can write

$$D = \sqrt{2^s + 1^{\alpha^2} + 1^{\alpha^4}} = \sqrt{(1^{\alpha^2} + 1^{\alpha^4})(1^{\alpha^2} + 1^{\alpha^4})} = 1^{\alpha^2} + 1^{\alpha^4}. \quad (54)$$

Fortunately, here the square root can be taken unambiguously, and the relative phase corresponding to the twofold degeneracy is  $\alpha^2$ . The other  $Z_6$  action arises through the permutation of the remaining two  $SU(3)$  subgroups. This gives four chiral fixed points, but as a rule, fixed points from permutation (sub-) orbifolds do not affect the degeneracy as their contribution is canceled against the volume factor of the invariant lattice. Thus from here cannot arise any *relative* phase.

Finally, we have to clarify whether there are some ambiguous *overall* phases in this sector, as was the case in the second twisted sector. The answer is that there are none, because any possible ambiguity can be resolved in the following way: untwisted and order two twisted sectors must by themselves be CPT invariant. That means that all phases must come in complex conjugate pairs. As shown, they already have this property so that there cannot be extra overall phases<sup>7</sup>.

Combining, finally, the internal phases with the NSR-phases we find that the positive helicity states are associated with  $Z_3$  phases 1 and  $\alpha^4$ . In the gauge part projections have to be made using  $\tilde{V}_3^{gauge} = 2(v_3, 0^4)$ . The factor 2 in the shift vector means that the obtained twist phases must match the *squares* of the phases shown in column  $S$  of table (44). Hence, we find matter transforming like

$$(78, 1, 1, 1, 1) + (1, 8, 1, 1, 1) + (27, 3, 1, 1, 1).$$

In contrast to the untwisted sector, from this sector can arise adjoint representations even if it is chiral.

The complete spectra of all models A through E are shown in Table 2.

## 5 $Z_2 \times Z_4$ orbifolds with $E_6$ and $SO(10)$ at level 2

In the course of the calculation in the last section, we encountered sign ambiguities which could not be resolved using standard asymmetric orbifold technology. In these cases, it was possible to fix the signs by simply requiring cancellation of anomalies, or by using other consistency arguments. In general, however, this information is insufficient. Moreover, in quite involved calculations one would rather reserve anomaly factorizations and cancellations as cross check.

The easiest way to resolve these ambiguities is to avoid twist rotations and to use instead space shifts leading to equivalent models<sup>8</sup>. Then all phases can be determined in a straightforward way, as will be worked out below in an example. I will present this example in considerable detail since there are many non-trivial phases arising in asymmetric orbifolds, which are unheard of in the symmetric case where most of them cancel between left and right movers.

Before launching the sample calculation, I first define eight models in the  $Z_2 \times Z_4$  orbifold class. Actually, each of these models grants the option of an extra discrete torsion sign [30] in the twist sector projectors. I will refer to (not) including this extra sign as negative (positive)

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<sup>7</sup>In general, there could still be an overall sign. For the present case, however, that would result in phases which are not third roots of unity, which are the only possible ones in a  $Z_3$  projection.

<sup>8</sup>The example of space shifting the  $Z_3 \times Z_3$  orbifold was given in Reference [29].

discrete torsion. Models I through IV have level 2 gauge group  $SO(10) \times SU(4)$  while models V through VIII have  $E_6 \times SU(2) \times U(1)$ . The level  $k$  of a  $U(1)$  is meaningful: it describes an absolute normalization in which only certain charges may appear for massless states; moreover, the gauge transformation of the antisymmetric tensor field (the duality transformed axion) is proportional to  $k$  which is important for the demonstration that a potential  $U(1)$  anomaly is canceled by the Green-Schwarz mechanism.

Consider first the  $Z_4$  suborbifold of models I through IV. When twisting all gauge coordinates of an  $E_8$  lattice by  $Z_4$ , there are 60 twist invariant orbits of the 240 roots corresponding to the gauge group  $SO(10) \times SU(4)$ . Twisting both  $E_8$  lattices in this way yields  $E_{vac}^{gauge} = 3/4$ . An equivalent gauge shift can be chosen to have the form

$$V_4^{gauge} = (V_4; V_4), \quad (55)$$

with

$$V_4 = (+\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, 0^5). \quad (56)$$

Given  $E_{vac}^{gauge} = 3/4$ , we have the options of twisting the left-handed space part by a four dimensional  $Z_2$  reflection (denoted  $Z'_2$  in the following), or not touching it at all. On the right hand side we have two choices of supersymmetric  $Z_4$  twists which are discussed below. They will be called  $Z_4$  and  $Z'_4$ . All these twists can be realized on  $SO(12)$  lattices. In summary, we have four possibilities to choose the twist eigenvalue structure:

$$\begin{array}{ll} \text{I} & ( Z'_2, Z_4 ), \\ \text{II} & ( \mathbf{1}, Z_4 ), \\ \text{III} & ( Z'_2, Z'_4 ), \\ \text{IV} & ( \mathbf{1}, Z'_4 ). \end{array} \quad (57)$$

Calling the right handed  $Z_4$  twist matrices  $\theta_4$  and  $\theta'_4$ , we define the explicit  $Z_2$  part of the  $Z_2 \times Z_4$  orbifold by  $\theta_2 = \theta_4^3 \theta'_4$ . The  $Z_2$  action on left-movers is again the outer automorphism of  $E_8 \times E_8$ .

Models V through VIII are defined in the same way and again on  $SO(12)$  lattices but with two modifications. The gauge shift vector is changed to

$$V_4 = (+\frac{1}{4}, +\frac{1}{4}, -\frac{1}{2}, 0^5), \quad (58)$$

and since now  $E_{vac}^{gauge} = 3/8$ , one must also change the left space twist to a reflection of either all six left coordinates ( $Z''_2$ ) or only two ( $Z'''_2$ ). Again there are four twist eigenvalue structures,

$$\begin{array}{ll} \text{V} & ( Z''_2, Z_4 ), \\ \text{VI} & ( Z'''_2, Z_4 ), \\ \text{VII} & ( Z''_2, Z'_4 ), \\ \text{VIII} & ( Z'''_2, Z'_4 ). \end{array} \quad (59)$$

The relevant NSR shift vectors  $v$  corresponding to the  $Z_4$ ,  $Z'_4$  and  $Z_2$  twists are

$$\begin{aligned} v_4 &= (+\frac{1}{4}, +\frac{1}{4}, -\frac{1}{2}, 0), \\ v'_4 &= (+\frac{1}{4}, -\frac{1}{4}, 0, 0), \\ v_4 - v'_4 &= v_2 = (0, +\frac{1}{2}, -\frac{1}{2}, 0). \end{aligned} \quad (60)$$

As discussed before, one is advised to consider shifts  $w$  in the  $SO(12)$  lattice which are equivalent to the twists introduced above, namely

$$\begin{aligned}
w_4 &= (-\tfrac{1}{4}, +\tfrac{1}{4}, -\tfrac{1}{2}, +\tfrac{1}{2}, 0, 0), \\
w'_4 &= (-\tfrac{1}{4}, -\tfrac{1}{4}, 0, +\tfrac{1}{2}, 0, 0), \\
w'_4 - w_4 &= w_2 = (0, -\tfrac{1}{2}, +\tfrac{1}{2}, 0, 0, 0), \\
w'_2 &= (-\tfrac{1}{2}, +\tfrac{1}{2}, 0, 0, 0, 0), \\
w''_2 &= (+\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2}, 0, 0, 0), \\
w'''_2 &= (+\tfrac{1}{2}, 0, 0, 0, 0, 0).
\end{aligned} \tag{61}$$

We may combine the left and right moving internal space parts, as well as the NSR-fermions into 16 dimensional vector spaces with Lorentzian signature  $(6, 10)$ . Then we can write space shift vectors for the  $Z_4$  and  $Z_2$  suborbifolds, respectively, as

$$\begin{aligned}
V_I &= (w'_2, w_4, v_4) & (-\tfrac{1}{2}), \\
V_{II} &= (0, w_4, v_4) & (-1), \\
V_{III} &= (w'_2, w'_4, v'_4) & (0), \\
V_{IV} &= (0, w'_4, v'_4) & (-\tfrac{1}{2}), \\
V_V &= (w''_2, w_4, v_4) & (-\tfrac{1}{4}), \\
V_{VI} &= (w'''_2, w_4, v_4) & (-\tfrac{3}{4}), \\
V_{VII} &= (w''_2, w'_4, v'_4) & (+\tfrac{1}{4}), \\
V_{VIII} &= (w'''_2, w'_4, v'_4) & (-\tfrac{1}{4}), \\
V_2 &= (0, w_2, v_2) & (-1),
\end{aligned} \tag{62}$$

where the  $Z_2$  is common to all eight models. I also displayed the length squares of these vectors with respect to the Lorentzian signature.  $V_2$  is chosen such that its scalar products with the other vectors vanish, thus avoiding extra complicated phases.

The resulting spectra are displayed in Tables 3 and 4. In the following, I will discuss the relevant points to compute model VI.

### Untwisted sector $(1, 1)$

The positive helicity ground states  $h$  with their shift phases  $e^{2\pi i h v}$  are

$\theta_R$	$Z_4$	$Z_2$
$(+\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2})$	1	1
$(+\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, +\tfrac{1}{2})$	$i$	1
$(-\tfrac{1}{2}, +\tfrac{1}{2}, -\tfrac{1}{2}, +\tfrac{1}{2})$	$i$	-1
$(-\tfrac{1}{2}, -\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2})$	-1	-1

(63)

Besides twist invariant adjoint representations of  $E_6 \times SU(2) \times U(1)$  one finds states transforming as (i)  $(\mathbf{27}, \mathbf{1})_{-2} + c.c.$ , (ii)  $(\mathbf{27}, \mathbf{2})_{+1} + (\mathbf{1}, \mathbf{2})_{-3}$ , and (iii)  $(\overline{\mathbf{27}}, \mathbf{2})_{-1} + (\mathbf{1}, \mathbf{2})_{+3}$ , with  $Z_4$  twist phases  $-1$ ,  $+i$  and  $-i$ , respectively.

The projection onto  $Z_2$  invariant states is simple, because one can always keep either the symmetric or the antisymmetric combination of two  $E_8$  vectors. Thus, we find the untwisted matter representations

$$2(\mathbf{27}, \mathbf{2})_{+1} + 2(\mathbf{1}, \mathbf{2})_{-3} + (\mathbf{27}, \mathbf{1})_{-2} + (\overline{\mathbf{27}}, \mathbf{1})_{+2},$$

which transform trivially under the enhanced gauge group  $G_6 = SO(10) \times U(1)$ . As can be seen from the table, extra matter transforming under  $G_6$  and invariant under  $Z_4$  (related to the last helicity state) does not survive  $Z_2$  projection.

#### First $Z_4$ twisted sector $(1, \theta_4)$

The only massless spinor in this sector is

$$p = (-\frac{1}{4}, -\frac{1}{4}, 0, +\frac{1}{2}). \quad (64)$$

The number of right and left fixed points is  $N_R^f = 16$  and  $N_L^f = 4$ , respectively, and  $\det g_{inv} = 4$  so that we find a degeneracy  $D = 4$ . In the shift description, however, the ground states are characterized by 8 right space vectors,

$$\begin{aligned} q_{1/2} &= (-\frac{1}{4}, +\frac{1}{4}, \mp\frac{1}{2}, \pm\frac{1}{2}, 0, 0), \\ q_{3/4} &= (-\frac{1}{4}, +\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{1}{2}, 0, 0), \\ q_{5/6} &= (+\frac{1}{4}, -\frac{1}{4}, 0, 0, \mp\frac{1}{2}, \pm\frac{1}{2}), \\ q_{7/8} &= (+\frac{1}{4}, -\frac{1}{4}, 0, 0, \pm\frac{1}{2}, \pm\frac{1}{2}), \end{aligned} \quad (65)$$

and they are correlated with the left movers. Indeed, in order to pass from  $q_{1/2}$  to  $q_{3/4}$  one must use an  $SO(12)$  vector in both, the left and right parts which corresponds to a root of  $SO(24)$ . Similarly, to pass from  $q_{1/2}$  to  $q_{5/6}$  one must use  $SO(12)$  spinors on both sides. This corresponds to the spinor congruency class in  $Spin(24)/Z_2$  which is the Euclideanized Narain lattice of the space part, i.e. the lattice with metric  $G$  from Eq. (17).

As for the gauge part we have to look for states satisfying

$$\frac{(P_{E_8 \times E_8} + V_4^{gauge})^2}{2} = \frac{7}{8} \text{ or } \frac{3}{8}. \quad (66)$$

Hence, before  $Z_2$  projection there are massless states

$$\begin{aligned} [(\mathbf{27}, \mathbf{1})_{-1/2}(\mathbf{1}, \mathbf{1})_{+3/2} + E_8 \leftrightarrow E_8 + (\mathbf{1}, \mathbf{2})_{-3/2}(\mathbf{1}, \mathbf{2})_{-3/2}] \otimes [(r_1, q_{1/2}, p) + (r_2, q_{3/4}, p)], \\ (\mathbf{1}, \mathbf{1})_{+3/2}(\mathbf{1}, \mathbf{1})_{+3/2} \otimes [(r_3, q_{1/2}, p) + (r_4, q_{3/4}, p) + (r_5, q_{5/6}, p) + (r_6, q_{7/8}, p)], \end{aligned} \quad (67)$$

where

$$\begin{aligned} r_{1/2} &= (\pm\frac{1}{2}, 0, 0, 0, 0, 0), \\ r_3 &= (-\frac{1}{2}, \pm 1, 0, 0, 0, 0), \\ r_4 &= (+\frac{1}{2}, \pm 1, 0, 0, 0, 0), \\ r_{5/6} &= (0, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}) \text{ (even/odd \# of - signs)}. \end{aligned} \quad (68)$$

The first line (67) shows the fourfold twist vacuum degeneracy. The second line comprises 104 states. In the twist formalism they would arise through two half-oscillator excitations and 24 weights ( $\mathbf{8}_v + \mathbf{8}_s + \mathbf{8}_c$ ) of an invariant  $SO(8)$  times the fourfold vacuum degeneracy.

For the  $Z_2$  projection, we first consider the phases  $e^{-2\pi i(q_{1/2}w_2 + pv_2)} = -e^{-2\pi i(q_{3/4}w_2 + pv_2)} = \mp 1$ . The important point here is that both signs have to be used giving rise to both the symmetric *and* antisymmetric combinations of  $E_8 \times E_8$ , and regardless of an extra discrete torsion sign there are the states transforming under  $E_6 \times SU(2) \times U(1) \times SO(10) \times U(1)$  like

$$[2(\mathbf{27}, \mathbf{1})_{+1} + (\mathbf{1}, \mathbf{3})_{-3} + (\mathbf{1}, \mathbf{1})_{-3}]\mathbf{1}_{\pm 1/2} + (\mathbf{1}, \mathbf{1})_{+3}\mathbf{10}_{\pm 1/2}.$$

But there are also the states involving  $q_{5/6}$  and  $q_{7/8}$ . For all of them we find  $e^{-2\pi i(qw_2 + pv_2)} = +1$ . That is, if there is no further torsion sign all these states survive and give rise to

$$2(\mathbf{1}, \mathbf{1})_{+3}[\mathbf{16}_0 + \overline{\mathbf{16}}_0],$$

but in case of negative torsion these states are completely projected out.

#### $Z_2 \times Z_4$ **twisted sector** $(\theta_2, \theta_4)$

Massless spinors in this sector are

$$p_{\pm} = (-\frac{1}{4}, +\frac{1}{4}, \mp\frac{1}{2}, \pm\frac{1}{2}), \quad (69)$$

and we have  $N_R^f = N_L^f = \det g_{inv} = 4$ , yielding  $D = 2$ . The corresponding ground states are characterized by

$$q_{1/2} = (-\frac{1}{4}, -\frac{1}{4}, 0, \pm\frac{1}{2}, 0, 0), \quad (70)$$

and gauge vectors (of the diagonal  $E_8$ ) must satisfy

$$\frac{(P_{E_8} + 2V_4)^2}{4} = \frac{3}{8}. \quad (71)$$

States satisfying the masslessness condition are (before  $Z_2$  projection)

$$[(\mathbf{27}, \mathbf{1})_{+1} + (\mathbf{1}, \mathbf{1})_{-3} + c.c.] \otimes [(r_1, q_1, p_{\pm}) + (r_2, q_2, p_{\pm})].$$

The  $Z_2$  projection requires great care. A look at the partition function reveals the following phases: (i) an overall minus sign from the left-handed vacuum energy,  $E_{vac}^{gauge} = 1/2$ , arising from the permutation of the two  $E_8$  factors, so that  $e^{2\pi i E_{vac}^{gauge}} = -1$ ; (ii) another overall minus sign from the space shift,  $e^{-i\pi V_2^2} = -1$ ; (iii) in case of negative discrete torsion, yet another overall minus sign; (iv) ground state contributions  $e^{-2\pi i(qw_2 + p_{\pm}v_2)} = \mp 1$  for both  $q$ ; and (v) another possible sign from  $e^{i\pi P_{E_8}^2/2}$  which contributes since the dual of the invariant (diagonal)  $E_8$  lattice is an odd lattice. The survivors are given by

$$\begin{aligned} & [(\mathbf{27}, \mathbf{1})_{+1} + (\mathbf{1}, \mathbf{1})_{-3}] \otimes [(r_1, q_1, p_+) + (r_2, q_2, p_+)], \\ & [(\overline{\mathbf{27}}, \mathbf{1})_{-1} + (\mathbf{1}, \mathbf{1})_{+3}] \otimes [(r_1, q_1, p_-) + (r_2, q_2, p_-)]. \end{aligned}$$

States carrying the negative helicity vector  $p_-$  must be complex conjugated. Thus we find

$$2[(\mathbf{27}, \mathbf{1})_{+1} + (\mathbf{1}, \mathbf{1})_{-3}],$$

while in case of negative torsion we would have the complex conjugate representations.

## Second $Z_4$ twisted sector $(1, \theta_4^2)$

This is the only twist sector without discrete torsion. The massless spinors are

$$p_{\pm} = (0, 0, \mp \frac{1}{2}, \pm \frac{1}{2}), \quad (72)$$

and we have  $N_R^f = 16$ ,  $N_L^f = 1$  and  $\det g_{inv} = 4$ , yielding  $D = 2$  and corresponding to

$$\begin{aligned} q_{1/2} &= (\mp \frac{1}{2}, \pm \frac{1}{2}, 0, 0, 0, 0), \\ q_{3/4} &= (\pm \frac{1}{2}, \pm \frac{1}{2}, 0, 0, 0, 0). \end{aligned} \quad (73)$$

These we have to combine with states satisfying

$$\frac{(P_{E_8 \times E_8} + 2V_4^{gauge})^2}{2} = 1 \text{ or } \frac{1}{2}, \quad (74)$$

and before further projections we have

$$\begin{aligned} &(\mathbf{1}, \mathbf{2})_0 [(\mathbf{27}, \mathbf{1})_{+1} + (\mathbf{1}, \mathbf{1})_{-3} + c.c.] \otimes (0, q_{3/4}, p_{\pm}), \\ &[(\mathbf{27}, \mathbf{1})_{+1} + (\mathbf{1}, \mathbf{1})_{-3} + c.c.] (\mathbf{1}, \mathbf{2})_0 \otimes (0, q_{3/4}, p_{\pm}), \\ &(\mathbf{1}, \mathbf{2})_0 (\mathbf{1}, \mathbf{2})_0 \otimes (r_{7/8}, q_{1/2}, p_{\pm}). \end{aligned}$$

In twist language,

$$\begin{aligned} r_7 &= (\pm 1, 0, 0, 0, 0, 0), \\ r_8 &= (0, \underline{\pm 1}, 0, 0, 0, 0), \end{aligned} \quad (75)$$

would be described by vector weights (the  $\mathbf{12}$ ) of  $SO(12)$ .

Next we have to perform the  $Z_4$  projection. It results a trivial overall twist phase due to  $e^{-2\pi i(V_{VI}^2 + V_4^{gauge2})} = +1$ . We just need to consider the positive helicity vector since this is a twist sector of order two and the negative helicity vector gives simply the CPT partners. The relevant phases are  $e^{-2\pi i(q_{1/2}w_4 + p_+v_4)} = \mp 1$ ,  $e^{-2\pi i(q_{3/4}w_4 + p_+v_4)} = -i$  and  $e^{2\pi i r_{7/8} w_2'''} = \mp 1$ . The  $Z_4$  survivors are given by

$$\begin{aligned} &(\mathbf{1}, \mathbf{2})_0 [(\mathbf{27}, \mathbf{1})_{+1} + (\mathbf{1}, \mathbf{1})_{-3}] \otimes (0, q_{3/4}, p_+), \\ &[(\mathbf{27}, \mathbf{1})_{+1} + (\mathbf{1}, \mathbf{1})_{-3}] (\mathbf{1}, \mathbf{2})_0 \otimes (0, q_{3/4}, p_+), \\ &(\mathbf{1}, \mathbf{2})_0 (\mathbf{1}, \mathbf{2})_0 \otimes [(r_7, q_1, p_+) + (r_8, q_2, p_+)]. \end{aligned}$$

Finally, we turn to the  $Z_2$  projection. Only the phase  $e^{-2\pi i(q_{1/2}w_2 + p_+v_2)} = \pm 1$  is of interest here which tells us to take the (anti)symmetric combination of  $E_8$  vectors for states involving  $r_7$  ( $r_8$ ). Hence, the contribution from this sector is

$$\begin{aligned} &2(\mathbf{27}, \mathbf{2})_{+1} \mathbf{1}_0 + 2(\mathbf{1}, \mathbf{2})_{-3} \mathbf{1}_0, \\ &(\mathbf{1}, \mathbf{3})_0 \mathbf{1}_{\pm 1} + (\mathbf{1}, \mathbf{1})_0 \mathbf{1}_0. \end{aligned}$$

$Z_2 \times Z_4^2$  **twisted sector**  $(\theta_2, \theta_4^2)$

Massless spinors in this sector are

$$p_{\pm} = (0, \mp \frac{1}{2}, 0, \pm \frac{1}{2}), \quad (76)$$

and again we have  $D = 2$  corresponding to

$$\begin{aligned} q_{1/2} &= (\mp \frac{1}{2}, 0, \pm \frac{1}{2}, 0, 0, 0), \\ q_{3/4} &= (\pm \frac{1}{2}, 0, \pm \frac{1}{2}, 0, 0, 0). \end{aligned} \quad (77)$$

These vectors have to be combined with the solutions of

$$\frac{P_{E_8}^2}{4} = \frac{1}{2} \text{ or } 0, \quad (78)$$

as well as with the half-integer oscillators. These states comprise the full **248** of  $E_8$  which has to be appropriately decomposed and combined with  $(0, q_{3/4}, p_{\pm})$ , and there is also the **1** to be combined with  $(r_{7/8}, q_{1/2}, p_{\pm})$ .

As for the  $Z_4$  projection, there is as in the previous sector the trivial contribution from  $e^{-2\pi i(V_{VI}^2 + V_4^{gauge^2})} = +1$ , but here is also a possible torsion sign. Using  $e^{-2\pi i(q_{3/4}w_4 + p_+v_4)} = -1/-i$  and the various twist phases of the states inside the **248** as in the untwisted sector, we find for positive torsion

$$\begin{aligned} &(\mathbf{27}, \mathbf{1})_{-2}\mathbf{1}_0 + c.c., \\ &(\mathbf{27}, \mathbf{2})_{+1}\mathbf{1}_0 + (\mathbf{1}, \mathbf{2})_{-3}\mathbf{1}_0, \end{aligned}$$

while for negative torsion we would have

$$\begin{aligned} &(\mathbf{78}, \mathbf{1})_0\mathbf{1}_0 + (\mathbf{1}, \mathbf{3})_0\mathbf{1}_0 + (\mathbf{1}, \mathbf{1})_0\mathbf{1}_0, \\ &(\mathbf{27}, \mathbf{2})_{-1}\mathbf{1}_0 + (\mathbf{1}, \mathbf{2})_{+3}\mathbf{1}_0. \end{aligned}$$

Since  $e^{-2\pi i(q_{1/2}w_4 + p_+v_4)} = i/1$  and  $e^{2\pi i(r_{7/8}w_2''')} = \mp 1$ , we may use the states involving  $q_2$  to infer for positive torsion extra states involving  $r_8$ ,

$$(\mathbf{1}, \mathbf{1})_0\mathbf{10}_0,$$

while for negative torsion we would have instead the ones involving  $r_7$ ,

$$(\mathbf{1}, \mathbf{1})_0\mathbf{1}_{\pm 1}.$$

$Z_2$  **twisted sector**  $(\theta_2, 1)$

This sector is very similar to the previous one, and the reader may follow the same steps when knowing that the only possible overall  $Z_4$  phase is the possible discrete torsion sign. For the model at hand, it turns that the states arising from this sector are identical to the ones from the previous sector.

## 6 Discussion

The model computed in section 5 with negative discrete torsion turns out to have four generations and two adjoint representations. Phenomenologically, supersymmetric four generation models are not strictly ruled out, if one allows one neutrino to be quite different and heavier compared to the others. Such models have been constructed in References [31].

The obtained spectra for  $E_6$  models from  $Z_2 \times Z_3$  orbifolds are summarized in Table 2. Surprisingly, models A and C, as well as models B and D turn out to be mirror models of each other. But the various states are rearranged between the different sectors, and in particular, the adjoint representations come from the untwisted sector in one model and from the third twisted (order two) sector in the mirror. This is interesting because it shows that it is irrelevant for phenomenology from which sector the adjoint Higgses arise. There is one model with two adjoint representations. It has vanishing net generation number without being non-chiral with respect to all gauge groups. There is one model with 18 generations, 6 antigerations and no exotic matter, which is an encouraging result as it shows that one can have adjoint representations without extra exotics also for groups different from  $E_6$ . Finally, there is one model with 9 generations, 3 antigerations and sextets of  $SU(3)$ . Although these models are related to the  $Z_6$  ( $Z'_6$ ) orbifold with 24 generations, we see that by using symmetric embedding and promoting it to level 2, the net generation number decreased by factors of two and four in the non-trivial cases.

The obtained  $SO(10)$  spectra are summarized in Table 3. Inspection of the Table shows that models II and IV with negative torsion are equivalent. Also model I with either torsion is equivalent to model III<sup>+</sup>, where the superscript denotes the torsion sign. The mirror model of those is given by model III<sup>-</sup>. Hence, there are 4 physically distinguishable models. These equivalences lead to an important observation: some of the adjoint representations arose as the twist survivors inside a **248** of  $E_8$ . Others resulted as the antisymmetric combination in the product of two vectors of  $SO(10)$ . The former are known to correspond to flat directions in the effective field theory. For example, if they are untwisted adjoint fields, then they are easily seen to correspond to continuous Wilson lines. But due to the equivalences just enumerated<sup>9</sup>, the same conclusion must hold also for the latter type of adjoints.

II<sup>+</sup> has 32 generations, 24 **10** and a **54** of  $SO(10)$ , but no adjoints. II<sup>-</sup> has 2 adjoints, 12 decouplets and a **54** but net generation number zero. IV<sup>+</sup> has even 4 adjoints and at the same time a **54**, but also vanishing net generation number, as well as no **10**. The most interesting case is represented by models I and III with 8 net generations, 2 adjoints, 22 decouplets and no **54**. However, none of the above spectra looks phenomenologically promising. Nevertheless, it is noteworthy that in this class of models the appearance of multiple adjoint representations is rather generic. This is to be compared with symmetric orbifolds where only one GUT Higgs field of  $SO(10)$ , either a **54** or **45** can be obtained [17].

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<sup>9</sup>Here I implicitly assume that the equivalences persist at the massive levels and also for the interactions. Although this seems reasonable for models constructed at maximally enhanced symmetry points, there is at least one example where two string models have the same massless spectra but differ at the massive levels: the heterotic  $SO(32)$  theory and Type I superstrings, where in the latter case the spinor class of  $SO(32)$  is perturbatively missing. Non-perturbatively, however, these theories are believed to be equivalent.

Cancellation of anomalies in the models of Tables 2 and 3 can be checked with help of the relation<sup>10</sup> for two-index symmetric representations of  $SU(N)$ ,

$$\text{tr}_{s^{ij}} F_{SU(N)}^3 = (N+4) \text{tr}_{SU(N)} F^3 \quad (N \geq 3), \quad (79)$$

which means for  $SU(3)$  ( $SU(4)$ ) that a **6** (**10**) representation contributes to the anomaly 7 (8) times the amount of a fundamental **3** (**4**).

The obtained  $E_6$  spectra from  $Z_2 \times Z_4$  orbifolds are summarized in Table 4.

Similar to the  $SO(10)$  models, model V with either torsion is equivalent to model VII<sup>-</sup>, but VII<sup>+</sup> is different. These models have zero net generation number, namely  $8+8$  and  $6+6$  generations, and are non-chiral. Models VI and VIII with negative torsion are equivalent, while model VIII<sup>+</sup> represents the mirror. These models are the most interesting ones as they have four net  $E_6$  generations ( $13+9$ ) and two adjoints. Again model VI<sup>+</sup> is different and has  $23+3$  generations, but no adjoint  $E_6$  matter.

Models VI and VIII have an anomalous  $U(1)$ . In general, the anomaly is given by

$$\begin{aligned} (2\pi)^2 I &= \frac{1}{48} \text{tr} R^2 \sum_{i,A} s_A^i (q_A^i F_A) - \frac{1}{6} \sum_{i,A} s_A^i (\text{tr}_{R^i} F_A^3) - \frac{1}{2} \sum_{i,j,A,B} s_{AB}^{ij} (\text{tr}_{R^i} F_A^2) (q_B^j F_B) \\ &- \sum_{i,j,k,A,B,C} s_{ABC}^{ijk} (q_A^i F_A) (q_B^j F_B) (q_C^k F_C), \end{aligned} \quad (80)$$

where  $s_A^i$  is the number of multiplets transforming in representation  $R^i$  (or with charge  $q_A^i$ ) under group factor  $G_A$ ,  $s_{AB}^{ij}$  is the number of multiplets transforming in representation  $(R^i, R^j)$  under  $G_A \times G_B$ , etc. The trace over curvature matrices in  $R$  is in the vector representation of  $SO(3,1)$ . The second term is the usual cubic anomaly, which must of course vanish for non-Abelian group factors; for Abelian factors  $\text{tr}_{R^i}$  has to be replaced by  $q_A^{i3}$  (and in the third term by  $q_A^{i2}$ ). Cancellation of anomalies then requires factorization into

$$(2\pi)^2 I = [\text{tr} R^2 - \sum_A k_A \alpha_A^{(1)} \text{Tr} F_A^2] \times [\sum_B \alpha_B^{(2)} F_B], \quad (81)$$

with

$$\alpha_A^{(1)} = \frac{1}{\tilde{h}_A}. \quad (82)$$

For  $U(1)$  factors (omitting the trace symbol)

$$\alpha_{U(1)}^{(1)} = N, \quad (83)$$

where  $N$  is defined through the level 1 relation,

$$h_{U(1)} = \frac{q^2}{N}, \quad (84)$$

and with the normalization suggested in Reference [24] one would choose  $N = 1$ . While at level one  $q^2$  is indeed directly related to the conformal dimension, at higher levels one can still

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<sup>10</sup>The symbol  $\text{tr}$  without specification refers to the trace in the fundamental representation, while  $\text{Tr}$  means trace in the adjoint.

use this relation when (like in the present case) the higher level  $U(1)$  factor can be traced back to a level one  $U(1)$ . Putting everything together one may write

$$\alpha_{U(1)}^{(2)} = \frac{1}{48} \sum_i q_i \quad (85)$$

and there is the condition

$$\frac{\sum_i q_i^3}{\sum_i q_i} = \frac{kN}{8}. \quad (86)$$

In the cases under consideration,  $N = 12$  and  $k = 2$  and Eq. (86) can be seen to be satisfied; moreover,  $\alpha_{U(1)}^{(2)} = 15$  (3) for model VI with positive (negative) torsion. The mixed gauge anomalies can now be checked using

$$\sum_{i,j} s_{A,U(1)}^{ij} q^i = 2k_A \alpha_A^{(1)} \alpha_{U(1)}^{(2)}, \quad (87)$$

where  $\alpha_A^{(1)}$  is given by Eq. (82) when working with traces in adjoint representations; when using fundamental representations, the  $\alpha_A^{(1)}$  are given by [24],

$$\begin{aligned} \alpha_{SU(N)}^{(1)} &= \alpha_{Sp(N)}^{(1)} = 2, \\ \alpha_{SO(N)}^{(1)} &= \alpha_{G_2}^{(1)} = 1, \quad (N \geq 5) \\ \alpha_{F_4}^{(1)} &= \alpha_{E_6}^{(1)} = \frac{1}{3}, \\ &\alpha_{E_7}^{(1)} = \frac{1}{6}, \\ &\alpha_{E_8}^{(1)} = \frac{1}{30}. \end{aligned} \quad (88)$$

Sometimes it is necessary to use Eq. (4). For example, for the present cases one needs

$$\text{tr}_{16} F_{SO(10)}^2 = 2 \text{tr} F_{SO(10)}^2. \quad (89)$$

Explicit relations are provided in the appendix of [24].

The net generation numbers of all 11 inequivalent models is even. The relative difficulty to obtain odd generation numbers has been noted before in the context of the free fermionic construction [9, 14, 15]. In level 1 orbifolds, it is known that turning on quantized Wilson lines can result in odd and, in particular, three generations [32]. The construction introduced in this paper possesses the option of turning on Wilson lines, as well, and this represents one of the possible generalizations. Another important generalization are models with levels  $k$  larger than 2, obtained by permuting  $k$  identical group factors. This way, one may obtain  $[SO(10)]^{k=3}$  models with a massless **120** multiplet. On the other hand, the fermionic construction allows only for levels of the form  $k = 2^n$  with  $n$  an integer. As mentioned in the introduction, one may also attempt to construct models with Standard Model gauge group at level 2 in order to improve coupling unification [12]. In the construction at hand, this requires to go to higher twist orders.

As a final spin off, the techniques developed in this work can be used even for known models at level  $k = 1$ : utilizing exclusively shifts as in Eqs. (62) for both, the space and gauge parts<sup>11</sup>, it is straightforward to compute correlation functions for the popular three generation models with quantized Wilson lines mentioned above [32]. Basically, one would only have to evaluate exponentials of the conformal field theory, similar to the torus case. In contrast, standard techniques [34] would require the calculation of twist field correlation functions which is rather involved. Moreover, in the presence of quantized Wilson lines, which as discussed in section 3 are related to asymmetric orbifolds, one would need the technology outlined in Reference [35]. This has not been carried out successfully, so that for the most interesting class of orbifolds interactions are presently unavailable.

To conclude, I have introduced a new approach to construct higher level string models. The construction is based on orbifolds which has the advantage that the models are exactly soluble and allow for exact deformations using orbifold moduli. For example, the untwisted adjoint Higgs fields can be represented as continuous Wilson line moduli. Moreover, using asymmetric twists it is possible to avoid the “one GUT Higgs theorem” valid in symmetric orbifolds with  $SO(10)$  gauge groups [17].

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<sup>11</sup>These are shifts acting in odd self-dual Lorentzian lattices with signature  $(22, 10)$ . Indeed, these models are explicit realizations of the “covariant lattices” as introduced in [33].

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	I	II	III	IV
$G_6$	$SO(8) \times SU(2)^2$	$SO(12)$	$SO(8) \times SU(2)^2$	$SO(12)$
$(1, 1)$	$(\mathbf{10}, \mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $2(\mathbf{16}, \mathbf{4}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(\mathbf{10}, \mathbf{6}, \mathbf{1})$ $2(\mathbf{16}, \mathbf{4}, \mathbf{1})$	$(\mathbf{45}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{15}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\mathbf{16}, \mathbf{4}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\overline{\mathbf{16}}, \overline{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(\mathbf{45}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{15}, \mathbf{1})$ $(\mathbf{16}, \mathbf{4}, \mathbf{1})$ $(\overline{\mathbf{16}}, \overline{\mathbf{4}}, \mathbf{1})$
$(1, \theta_4)$	$(\mathbf{1}, \mathbf{6}, \mathbf{1}, \mathbf{2}, \mathbf{1})$ $(\mathbf{1}, \overline{\mathbf{10}}, \mathbf{1}, \mathbf{2}, \mathbf{1})$	$2(\mathbf{16}, \overline{\mathbf{4}}, \mathbf{1})$	$(\mathbf{10}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$ $(\mathbf{1}, \mathbf{6}, \mathbf{1}, \mathbf{2}, \mathbf{1})$	$(\mathbf{16}, \overline{\mathbf{4}}, \mathbf{1})$ $(\overline{\mathbf{16}}, \mathbf{4}, \mathbf{1})$
$(\theta_2, \theta_4)$	$(\mathbf{10}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$ $(\mathbf{1}, \mathbf{6}, \mathbf{1}, \mathbf{2}, \mathbf{1})$	$[2(\mathbf{16}, \overline{\mathbf{4}}, \mathbf{1})]^+$ $[2(\overline{\mathbf{16}}, \mathbf{4}, \mathbf{1})]^-$	$(\mathbf{1}, \mathbf{6}, \mathbf{1}, \mathbf{2}, \mathbf{1})$ $[(\mathbf{1}, \mathbf{10}, \mathbf{1}, \mathbf{2}, \mathbf{1})]^+$ $[(\mathbf{1}, \overline{\mathbf{10}}, \mathbf{1}, \mathbf{2}, \mathbf{1})]^-$	$(\mathbf{16}, \overline{\mathbf{4}}, \mathbf{1})$ $(\overline{\mathbf{16}}, \mathbf{4}, \mathbf{1})$
$(1, \theta_4^2)$	$(\mathbf{45}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{15}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\mathbf{10}, \mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(\mathbf{54}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{20}', \mathbf{1})$ $2(\mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\mathbf{10}, \mathbf{6}, \mathbf{1})$	$2(\mathbf{10}, \mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(\mathbf{54}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{20}', \mathbf{1})$ $2(\mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\mathbf{45}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{15}, \mathbf{1})$
$(\theta_2, \theta_4^2)^+$	$(\mathbf{45}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{15}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\overline{\mathbf{16}}, \overline{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2})$	$(\mathbf{10}, \mathbf{6}, \mathbf{1})$ $(\mathbf{16}, \mathbf{4}, \mathbf{1})$ $(\mathbf{1}, \mathbf{1}, \mathbf{12})$	$(\mathbf{10}, \mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\overline{\mathbf{16}}, \overline{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2})$	$(\mathbf{45}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{15}, \mathbf{1})$ $(\mathbf{16}, \mathbf{4}, \mathbf{1})$ $(\mathbf{1}, \mathbf{1}, \mathbf{12})$
$(\theta_2, \theta_4^2)^-$	$(\mathbf{10}, \mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\mathbf{16}, \mathbf{4}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{1}, \mathbf{8}, \mathbf{1}, \mathbf{1})$	$(\mathbf{45}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{15}, \mathbf{1})$ $(\overline{\mathbf{16}}, \overline{\mathbf{4}}, \mathbf{1})$	$(\mathbf{45}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{15}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\mathbf{16}, \mathbf{4}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{1}, \mathbf{8}, \mathbf{1}, \mathbf{1})$	$(\mathbf{10}, \mathbf{6}, \mathbf{1})$ $(\overline{\mathbf{16}}, \overline{\mathbf{4}}, \mathbf{1})$
$(\theta_2, 1)^+$	same as $(\theta_2, \theta_4^2)^-$	same as $(\theta_2, \theta_4^2)^+$	<i>c.c.</i> of $(\theta_2, \theta_4^2)^-$	<i>c.c.</i> of $(\theta_2, \theta_4^2)^+$
$(\theta_2, 1)^-$	same as $(\theta_2, \theta_4^2)^+$	same as $(\theta_2, \theta_4^2)^-$	<i>c.c.</i> of $(\theta_2, \theta_4^2)^+$	<i>c.c.</i> of $(\theta_2, \theta_4^2)^-$

Table 3: Models from asymmetric  $Z_2 \times Z_4$  orbifolds with gauge group  $[SO(10) \times SU(4)]^{k=2} \times G_6^{k=1}$ . Superscripts  $\pm$  refer to positive and negative discrete torsion.

	V	VI	VII	VIII
$G_6$	$SU(4) \times SU(4)$	$SO(10) \times U(1)$	$SU(4) \times SU(4)$	$SO(10) \times U(1)$
$(1, 1)$	$(\mathbf{27}, \mathbf{1})_{-2} + c.c.$ $2(\mathbf{27}, \mathbf{2})_{+1}$ $2(\mathbf{1}, \mathbf{2})_{-3}$	$(\mathbf{27}, \mathbf{1})_{-2} + c.c.$ $2(\mathbf{27}, \mathbf{2})_{+1}$ $2(\mathbf{1}, \mathbf{2})_{-3}$	$(\mathbf{78}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{3})_0 + (\mathbf{1}, \mathbf{1})_0$ $(\mathbf{27}, \mathbf{2})_{+1} + c.c.$ $(\mathbf{1}, \mathbf{2})_{\pm 3}$	$(\mathbf{78}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{3})_0 + (\mathbf{1}, \mathbf{1})_0$ $(\mathbf{27}, \mathbf{2})_{+1} + c.c.$ $(\mathbf{1}, \mathbf{2})_{\pm 3}$
$(1, \theta_4)$	$2(\mathbf{1}, \mathbf{2})_0(4 + \overline{4}, \mathbf{1})$ $2(\mathbf{1}, \mathbf{2})_0(\mathbf{1}, 4 + \overline{4})$	$2(\mathbf{27}, \mathbf{1})_{+1}\mathbf{1}_{\pm 1/2}$  $(\mathbf{1}, \mathbf{3} + \mathbf{1})_{-3}\mathbf{1}_{\pm 1/2}$ $(\mathbf{1}, \mathbf{1})_{+3}\mathbf{10}_{\pm 1/2}$ $[2(\mathbf{1}, \mathbf{1})_{+3}\mathbf{16}_0]^+$ $[2(\mathbf{1}, \mathbf{1})_{+3}\overline{\mathbf{16}}_0]^+$	$(\mathbf{1}, \mathbf{2})_0(4 + \overline{4}, \mathbf{1})$ $[2(\mathbf{1}, \mathbf{2})_0(\mathbf{1}, 4 + \overline{4})]^-$	$(\mathbf{27}, \mathbf{1})_{+1}\mathbf{1}_{\pm 1/2} + c.c.$  $(\mathbf{1}, \mathbf{1})_{-3}\mathbf{1}_{\pm 1/2} + c.c.$
$(\theta_2, \theta_4)^+$	$(\mathbf{1}, \mathbf{2})_0(4 + \overline{4}, \mathbf{1})$	$2(\mathbf{27}, \mathbf{1})_{+1}\mathbf{1}_{\pm 1/2}$  $2(\mathbf{1}, \mathbf{1})_{-3}\mathbf{1}_{\pm 1/2}$	$2(\mathbf{1}, \mathbf{2})_0(4 + \overline{4}, \mathbf{1})$	$(\mathbf{27}, \mathbf{1})_{+1}\mathbf{1}_{\pm 1/2} + c.c.$  $(\mathbf{1}, \mathbf{3})_{+3}\mathbf{1}_{\pm 1/2}$ $(\mathbf{1}, \mathbf{1})_{-3}\mathbf{1}_{\pm 1/2}$ $(\mathbf{1}, \mathbf{1})_{-3}\mathbf{10}_{\pm 1/2}$
$(\theta_2, \theta_4)^-$	same as $(\theta_2, \theta_4)^+$	<i>c.c.</i> of $(\theta_2, \theta_4)^+$	same as $(\theta_2, \theta_4)^+$	<i>c.c.</i> of $(\theta_2, \theta_4)^+$
$(1, \theta_4^2)$	$2(\overline{\mathbf{27}}, \mathbf{2})_{-1}$ $2(\mathbf{1}, \mathbf{2})_{+3}$ $(\mathbf{1}, \mathbf{3})_0(\mathbf{1}, \mathbf{6})$ $(\mathbf{1}, \mathbf{1})_0(\mathbf{6}, \mathbf{1})$	$2(\mathbf{27}, \mathbf{2})_{+1}$ $2(\mathbf{1}, \mathbf{2})_{-3}$ $(\mathbf{1}, \mathbf{3})_0\mathbf{1}_{\pm 1}$ $(\mathbf{1}, \mathbf{1})_0\mathbf{10}_0$	$(\mathbf{27}, \mathbf{2})_{+1} + c.c.$ $(\mathbf{1}, \mathbf{2})_{\pm 3}$ $(\mathbf{1}, \mathbf{3})_0(\mathbf{1}, \mathbf{6})$ $(\mathbf{1}, \mathbf{1})_0(\mathbf{1}, \mathbf{6})$	$(\mathbf{27}, \mathbf{2})_{+1} + c.c.$ $(\mathbf{1}, \mathbf{2})_{\pm 3}$ $(\mathbf{1}, \mathbf{3})_0\mathbf{1}_{\pm 1}$ $(\mathbf{1}, \mathbf{1})_0\mathbf{1}_{\pm 1}$
$(\theta_2, \theta_4^2)^+$	$(\mathbf{78}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{3})_0 + (\mathbf{1}, \mathbf{1})_0$ $(\mathbf{27}, \mathbf{2})_{-1}$ $(\mathbf{1}, \mathbf{2})_{+3}$ $(\mathbf{1}, \mathbf{1})_0(\mathbf{6}, \mathbf{1})$	$(\mathbf{27}, \mathbf{1})_{-2} + c.c.$ $(\mathbf{27}, \mathbf{2})_{+1}$ $(\mathbf{1}, \mathbf{2})_{-3}$ $(\mathbf{1}, \mathbf{1})_0\mathbf{10}_0$	$(\mathbf{78}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{3})_0 + (\mathbf{1}, \mathbf{1})_0$ $(\mathbf{27}, \mathbf{2})_{+1}$ $(\mathbf{1}, \mathbf{2})_{-3}$ $(\mathbf{1}, \mathbf{1})_0(\mathbf{1}, \mathbf{6})$	$(\mathbf{27}, \mathbf{1})_{-2} + c.c.$ $(\mathbf{27}, \mathbf{2})_{-1}$ $(\mathbf{1}, \mathbf{2})_{+3}$ $(\mathbf{1}, \mathbf{1})_0\mathbf{1}_{\pm 1}$
$(\theta_2, \theta_4^2)^-$	$(\mathbf{27}, \mathbf{1})_{-2} + c.c.$ $(\mathbf{27}, \mathbf{2})_{+1}$ $(\mathbf{1}, \mathbf{2})_{-3}$ $(\mathbf{1}, \mathbf{1})_0(\mathbf{1}, \mathbf{6})$	$(\mathbf{1}, \mathbf{3})_0 + (\mathbf{1}, \mathbf{1})_0$ $(\mathbf{27}, \mathbf{2})_{-1}$ $(\mathbf{1}, \mathbf{2})_{+3}$ $(\mathbf{1}, \mathbf{1})_0\mathbf{1}_{\pm 1}$	$(\mathbf{27}, \mathbf{1})_{-2} + c.c.$ $(\mathbf{27}, \mathbf{2})_{-1}$ $(\mathbf{1}, \mathbf{2})_{+3}$ $(\mathbf{1}, \mathbf{1})_0(\mathbf{6}, \mathbf{1})$	$(\mathbf{78}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{3})_0 + (\mathbf{1}, \mathbf{1})_0$ $(\mathbf{27}, \mathbf{2})_{+1}$ $(\mathbf{1}, \mathbf{2})_{-3}$ $(\mathbf{1}, \mathbf{1})_0\mathbf{10}_0$
$(\theta_2, 1)^+$	same as $(\theta_2, \theta_4^2)^-$	same as $(\theta_2, \theta_4^2)^+$	<i>c.c.</i> of $(\theta_2, \theta_4^2)^+$	<i>c.c.</i> of $(\theta_2, \theta_4^2)^-$
$(\theta_2, 1)^-$	same as $(\theta_2, \theta_4^2)^+$	same as $(\theta_2, \theta_4^2)^-$	<i>c.c.</i> of $(\theta_2, \theta_4^2)^-$	<i>c.c.</i> of $(\theta_2, \theta_4^2)^+$

Table 4: Models from asymmetric  $Z_2 \times Z_4$  orbifolds with gauge group  $[E_6 \times SU(2) \times U(1)]^{k=2} \times G_6^{k=1}$ . Superscripts are as in Table 3. Only non-trivial representations of  $G_6$  are shown.